MATH2060B Tutorial 2

Topic: Mean value theorem

<u>Recall</u>: Suppose f is continuous on $[a,b]$ and differentiable on (a,b) $\Rightarrow \exists c \in (a,b)$ s.t. $f(b)-f(a) = f'(c)(b-a)$.

<u>Q6</u>. Show that $|\sin x - \sin y| \le |x-y| \quad \forall x,y \in \mathbb{R}$

$\rightsquigarrow$ use $\sin x$ differentiable $\forall x$, with derivative $\cos x$

<u>Sol</u>. $x=y \rightsquigarrow$ trivial

$x \ne y \rightsquigarrow$ Apply MVT to $[x,y]$ (or $[y,x]$), and $\sin$.

$\Rightarrow |\sin x - \sin y| = |(\cos c)||(x-y)|$ for some c between x, y

$\le |x-y| \quad \because |\cos c| \le 1 \; \forall c \in \mathbb{R}$ //

<u>Q8</u> Let $f:[a,b] \to \mathbb{R}$ be cts and differentiable on (a,b). Show that if $\lim_{x\to a^+} f'(x) = A$, then $f'(a)$ exists and $=A$.

Want to show $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ exists. (cf. L'hopital)

<u>Pf</u>. Condition $\lim_{x\to a^+} f'(x) = A$ implies with $\varepsilon > 0$ fixed, $\exists \delta > 0$ s.t. $\forall x \in (a, a+\delta)$, (with $a+\delta < b$) we have $|f'(x) - A| < \varepsilon$.

For $x \in (a, a+\delta)$, MVT applied to $[a,x]$ (f diff. on (a,x), f cts at a!)

$$f(x) - f(a) = f'(c)(x-a) \quad c \in (a,x) \subset (a,a+\delta)$$

$$\Rightarrow \left| \frac{f(x)-f(a)}{x-a} - A \right| = |f'(c) - A| < \varepsilon.$$

$\because$ holds $\forall x \in (a, a+\delta)$, $\varepsilon > 0$ arbitrary $\Rightarrow f'(a)$ exists and equals A. $\square$

right derivative

$\rightsquigarrow$ Similar results hold for left / two-sided derivatives.

$\rightsquigarrow$ useful criterion for differentiability: Suppose f is cts at a and differentiable in an open interval of a (except a)

$\Rightarrow$ if $\lim_{x\to a} f'(x)$ exists, then $f'(a)$ also exists.

Q15. Let I be an interval, $f: I \to \mathbb{R}$ is differentiable with bounded derivative $f': I \to \mathbb{R}$. Show that f satisfies a Lipschitz condition on I.

Recall a function $g: I \to \mathbb{R}$ is Lipschitz if $\exists K > 0$ s.t.
$|f(x) - f(y)| \leq K|x-y| \quad \forall x, y \in I$.

Pf Set K to be a bound of f', i.e. $|f'(p)| \leq K \ \forall p \in I$.

To show K satisfies Lipschitz ineq.

$x = y \rightsquigarrow$ trivial

$x \neq y \rightsquigarrow$ Apply MVT to $[x,y]$ (or $[y,x]$)

$$\Rightarrow |f(x) - f(y)| = |f'(c)||x-y| \quad \text{for } c \in I \text{ (between } x \text{ and } y)$$
$$\leq K|x-y| \qquad \square$$

Q17 Let f, g differentiable on $\mathbb{R}$. $f(0) = g(0)$, $f'(x) \leq g'(x) \ \forall x \geq 0$
Show $f(x) \leq g(x) \ \forall x \geq 0$.

Sol For $x \geq 0$, apply MVT to $g - f$ on $[0, x]$ ← differentiable & $(g-f)' = g' - f' \geq 0$ on $[0,x]$

$$\Rightarrow (g-f)(x) - (g-f)(0) = (g-f)'(c) \cdot (x - 0), \quad c \in (0,x)$$
$$g(x) - f(x) - \cancel{g(0) + f(0)} \geq 0$$
$$g(x) \geq f(x) \qquad \forall x \in [0, \infty) \ //$$

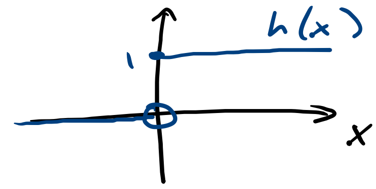
Q14 Let I be an interval, $f: I \to \mathbb{R}$ differentiable.
Show that if f' is never zero, then either $f'(x) > 0$ or or $f'(x) < 0 \ \forall x \in I$.

Recall: Darboux's Thm: derivative f' of a differentiable $f: [a,b] \to \mathbb{R}$, satisfies: $\forall k$, $f'(a) < k < f'(b)$, $\exists c \in (a,b)$ s.t. $f'(c) = k$.

In particular, if $f: I \to \mathbb{R}$ differentiable, and there exist points $x, y \in I$ s.t. $f'(x) > 0$, $f'(y) < 0$ $\Rightarrow$ Darboux's Thm applied to $[x,y]$ (or $[y,x]$) gives $c \in I$, $f'(c) = 0$

$\rightsquigarrow$ contrapositive of the statement in q.14.

Q12 Let $h(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. Prove that there does not exist $f: \mathbb{R} \to \mathbb{R}$ s.t. $f'(x) = h(x)$ $\forall x \in \mathbb{R}$.

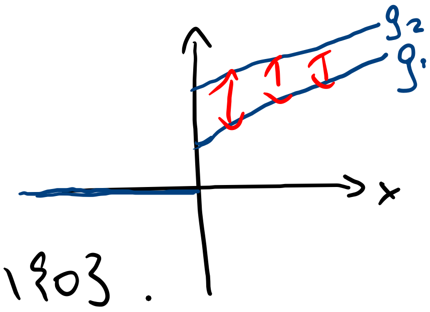


Pf If such f exists, then $f'(-1) = \underline{0}$, and $f'(1) = \underline{1}$. Apply Darboux's Thm to f & $[-1, 1]$, gives $c \in (-1, 1)$ s.t. $f'(c) = 1/2$

$\because 0 < \underline{\underline{1/2}} < 1$

$\underset{h(c)}{\overset{f'(c)}{\|}}$ contradiction $\square$

☆ Can define $g_1, g_2 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ (for $i = 1, 2$). $g_i(x) = \begin{cases} x + i, & x > 0 \\ 0 & x < 0 \end{cases}$



$\Rightarrow g_1'(x) = g_2'(x) = h(x)$ $\forall x \in \mathbb{R} \setminus \{0\}$.

Remark ① Do not contradict Darboux's Thm since $\mathbb{R} \setminus \{0\}$ is the union of two disjoint open intervals $(0, \infty)$ and $(-\infty, 0)$

② $g_1 - g_2$ is NOT a constant

$\therefore$ This is a counterexample to Cor. 6.2.6. when domain is not an interval.

i.e. "anti-derivatives are unique up to a constant" only when functions are defined on an interval.

(Not covered in actual tutorial)

16c. Let $f:[0,\infty)\to\mathbb{R}$ be differentiable on $(0,\infty)$ and assume that $f'(x)\to b$ as $x\to\infty$. Show that $\lim_{x\to\infty}\frac{f(x)}{x}=b$.

$\rightsquigarrow$ Preview / review of L'hopital's rule

Pf Given condition $f'(x)\to b$ as $x\to\infty$. $\Rightarrow \forall\varepsilon>0, \exists K>0$ s.t. if $x>K$, then $|f'(x)-b|<\varepsilon$. For $x>K$, write

$$\frac{f(x)}{x}=\frac{f(x)-f(K)}{x-K}\cdot\left(\frac{x-K}{x}\right)+\frac{f(K)}{x}$$

$$\overset{MVT}{=} f'(c_x)\cdot 1-\frac{Kf'(c_x)}{x}+\frac{f(K)}{x}$$

for some $c_x>K$. (in fact $c_x<x$)

$\rightsquigarrow$ for $x>K$, $\left|\frac{f(x)}{x}-b\right|\le|f'(c_x)-b|+\left|\frac{-Kf'(c_x)+f(K)}{x}\right|$

$$<\varepsilon+\left|\frac{-Kf'(c_x)+f(K)}{x}\right|$$

Notice that $-Kf'(c_x)+f(K)$ is bounded (indep. of x) $\because c_x>K\ \forall x>K$

$\Rightarrow$ choose $K'>K$ s.t. $\forall x>K'$,

$$\left|\frac{-Kf'(c_x)+f(K)}{x}\right|<\varepsilon.$$

$\Rightarrow$ for $x>K'>K$,

$$\left|\frac{f(x)}{x}-b\right|<\varepsilon+\varepsilon=2\varepsilon. \qquad \square$$